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Doubly Periodic Wave Solutions and Soliton Solutions of Ablowitz–Ladik Lattice System

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Abstract The general Jacobi elliptic function expansion method is developed and extended to construct doubly periodic wave solutions for discrete nonlinear equations. Applying this method, many exact elliptic function doubly periodic wave solutions are obtained for Ablowitz–Ladik lattice system. When the modulus $m \to 1$ or $m \to 0$, these solutions degenerate into hyperbolic function solutions and trigonometric function solutions respectively. In long wave limit, solitonic solutions including bright soliton and dark soliton solutions are also obtained.

Keywords Ablowitz–Ladik system · Jacobi elliptic function · Soliton

1 Introduction

In nonlinear science, the study of differential-difference equations (DDEs) especially the integrable lattice equations has aroused increasing interest. The integrable lattice equations may exhibit self-localized excitations in form of solitons or breathers. Discrete solitons, which are intrinsic highly localized models of nonlinear lattice [1] that form when discrete diffraction is balanced by nonlinearity, have been demonstrated to exist in a wide range of discrete physical systems, such as biological systems [2], atomic chains [3, 4], molecular crystals [5], electrical lattices [6], Bose–Einstein condensates [7], photonic structures [8, 9], etc. Seeking and the investigation of exact discrete solutions especially soliton solutions of DDEs play an important role in the study of discrete nonlinear physical system. However, it is more difficult to find exact solutions for DDEs than that for continuum nonlinear system actually. In recent years, many powerful approaches are presented to find and investigate exact discrete solutions for DDES. Especially, with the development of computer

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symbolic computation, many direct and effective algebraic methods that are presented firstly for solving continuous nonlinear partial differential equations (NPDEs) are developed and successfully extended to construct exact discrete solutions of DDEs, such as the multi-linear variable separation approach [10, 11], tanh function expansion method [12, 13], the hyperbolic function approach [14], the sine-Gordon expansion method [15], the Jacobi elliptic function expansion method [16–19], etc.

In this paper, based on the general Jacobi elliptic function expansion method for NPDEs presented by Dazhao Lü [20], we developed this method to solve DDES and construct exact discrete solutions. Applying this method to Ablowitz–Ladik (AL) lattice system [21], we derived abundant Jacobian elliptic function doubly periodic wave solutions. When the modulus of elliptic functions $m \to 1$ or 0, solitonic solutions including bright soliton and dark soliton solutions are generated.

Our paper is organized as follows. In Sect. 2, a detailed description of the proposed method will be given. In Sect. 3, the application of the proposed method to the AL system is illustrated. We derive many Jacobi elliptic function doubly periodic wave solutions and give brief discussions for their degenerated solutions especially the soliton solutions. The final section is a short summary.

2 General Jacobi Elliptic Function Expansion Method for DDES

For a given system of M polynomial DDES

$$\Delta(u_{n+p_1}(x), \dots, u_{n+p_\tau}(x), \dots, u'_{n+p_1}(x), \dots, u'_{n+p_\tau}(x), \dots, u^{(r)}_{n+p_1}(x), \dots, u^{(r)}_{n+p_\tau}(x)) = 0, \tag{1}$$

where the depended variable u has M components u_{in}, the continuous variable x has N components x_i, the discrete variable n has Q components n_j, the τ shift vectors p_i, and $u^{(r)}(x)$ denote the collection of mixed derivative terms of order r. The main steps of the general Jacobian elliptic function method are outlined as follows.

Step 1 When we seek the traveling wave solutions of (1), the first step is to introduce the wave transformation $u_{n+p_s}(x) = \phi_{n+p_s}(\xi_n)$, $\xi_n = \sum_{i=1}^{Q} d_i n_i + \sum_{j=1}^{N} c_j x_j + \zeta$ for any s $(s = 1, \dots, \tau)$, where the coefficients $c_1, c_2, \dots, c_N, d_1, d_2, \dots, d_Q$ and the phase ζ are all constants. In this way, (1) becomes

$$\begin{aligned}&\Delta(u_{n+p_1}(\xi_n), \dots, u_{n+p_k}(\xi_n), \dots, u'_{n+p_1}(\xi_n), \dots, u'_{n+p_k}(\xi_n), \dots,\\ &u^{(r)}_{n+p_1}(\xi_n), \dots, u^{(r)}_{n+p_k}(\xi_n)) = 0,\end{aligned} \tag{2}$$

Step 2 To seek the doubly periodic solutions, we assume (1) or (2) has a solution expressed by the following series expansion

$$\begin{aligned}\phi_n(\xi_n) = {}& a_0 + a_1 f_{n,i}(\xi_n) + b_1 g_{n,i}(\xi_n) + c_1 h_{n,i}(\xi_n) + \sum_{j=2}^{l} f_{n,i}^{j-2}(\xi_n)[a_j f_{n,i}^2(\xi_n)\\ &+ b_j f_{n,i}(\xi_n) g_{n,i}(\xi_n) + c_j f_{n,i}(\xi_n) h_{n,i}(\xi_n) + d_j g_{n,i}(\xi_n) h_{n,i}(\xi_n)] \quad (i = 1, 2, 3, 4)\end{aligned} \tag{3}$$

with

$$f_{n,1}(\xi_n) = \operatorname{sn}\xi_n, \qquad g_{n,1}(\xi_n) = \operatorname{cn}\xi_n, \qquad h_{n,1}(\xi_n) = \operatorname{dn}\xi_n,$$

$$
\begin{aligned}
&f_{n,2}(\xi_n)=\operatorname{ns}\xi_n=\frac{1}{\operatorname{sn}\xi_n}, \qquad g_{n,2}(\xi_n)=\operatorname{cs}\xi_n=\frac{\operatorname{cn}\xi_n}{\operatorname{sn}\xi_n}, \qquad h_{n,2}(\xi_n)=\operatorname{ds}\xi_n=\frac{\operatorname{dn}\xi_n}{\operatorname{sn}\xi_n},\\
&f_{n,3}(\xi_n)=\operatorname{sc}\xi_n=\frac{\operatorname{sn}\xi_n}{\operatorname{cn}\xi_n}, \qquad g_{n,3}(\xi_n)=\operatorname{nc}\xi_n=\frac{1}{\operatorname{cn}\xi_n}, \qquad h_{n,3}(\xi_n)=\operatorname{dc}\xi_n=\frac{\operatorname{dn}\xi_n}{\operatorname{cn}\xi_n},\\
&f_{n,4}(\xi_n)=\operatorname{sd}\xi_n=\frac{\operatorname{sn}\xi_n}{\operatorname{dn}\xi_n}, \qquad g_{n,4}(\xi_n)=\operatorname{cd}\xi_n=\frac{\operatorname{cn}\xi_n}{\operatorname{dn}\xi_n}, \qquad h_{n,4}(\xi_n)=\operatorname{nd}\xi_n=\frac{1}{\operatorname{dn}\xi_n},
\end{aligned}
\tag{4}
$$

where l is the integer to be determined later, while $\operatorname{sn}\xi_n=\operatorname{sn}(\xi_n,m)$, $\operatorname{cn}\xi_n=\operatorname{cn}(\xi_n,m)$, $\operatorname{dn}\xi_n=\operatorname{dn}(\xi_n,m)$ with the modulus m $(0<m<1)$ are the Jacobian elliptic sine function, the Jacobian cosine function and the Jacobian elliptic function of the third kind respectively. The other Jacobian functions, which are denoted by Glaisher's symbols, are generated by these three kinds of functions, and there are the relations

$$
\begin{aligned}
&\operatorname{cn}^2(\xi_n)=-\operatorname{sn}^2(\xi_n)+1, \qquad \operatorname{dn}^2(\xi_n)=-m^2\operatorname{sn}^2(\xi_n)+1,\\
&\operatorname{dn}^2(\xi_n)=m^2\operatorname{cn}^2(\xi_n)+(1-m^2), \qquad \operatorname{ns}^2(\xi_n)=\operatorname{cs}^2(\xi_n)+1,\\
&\operatorname{ns}^2(\xi_n)=\operatorname{ds}^2(\xi_n)+m^2, \qquad \operatorname{ds}^2(\xi_n)=\operatorname{cs}^2(\xi_n)+(1-m^2),\\
&\operatorname{nc}^2(\xi_n)=\operatorname{sc}^2(\xi_n)+1, \qquad \operatorname{dc}^2(\xi_n)=(1-m^2)\operatorname{nc}^2(\xi_n)+m^2,\\
&\operatorname{dc}^2(\xi_n)=(1-m^2)\operatorname{sc}^2(\xi_n)+1, \qquad \operatorname{cd}^2(\xi_n)=\frac{m^2-1}{m^2}\operatorname{nd}^2(\xi_n)+\frac{1}{m^2},\\
&\operatorname{cd}^2(\xi_n)=(m^2-1)\operatorname{sd}^2(\xi_n)+1, \qquad \operatorname{nd}^2(\xi_n)=m^2\operatorname{sd}^2(\xi_n)+1,
\end{aligned}
$$

and

$$
\begin{aligned}
&(\operatorname{sn}\xi_n)'=\operatorname{cn}\xi_n\operatorname{dn}\xi_n, \qquad (\operatorname{cn}\xi_n)'=-\operatorname{sn}\xi_n\operatorname{dn}\xi_n, \qquad (\operatorname{dn}\xi_n)'=-m^2\operatorname{sn}\xi_n\operatorname{cn}\xi_n,\\
&(\operatorname{ns}\xi_n)'=-\operatorname{cs}\xi_n\operatorname{ds}\xi_n, \qquad (\operatorname{cs}\xi_n)'=-\operatorname{ns}\xi_n\operatorname{ds}\xi_n, \qquad (\operatorname{ds}\xi_n)'=-\operatorname{ns}\xi_n\operatorname{cs}\xi_n,\\
&(\operatorname{sc}\xi_n)'=\operatorname{nc}\xi_n\operatorname{dc}\xi_n, \qquad (\operatorname{nc}\xi_n)'=\operatorname{sc}\xi_n\operatorname{dc}\xi_n, \qquad (\operatorname{dc}\xi_n)'=(1-m^2)\operatorname{nc}\xi_n\operatorname{sc}\xi_n,\\
&(\operatorname{sd}\xi_n)'=\operatorname{nd}\xi_n\operatorname{cd}\xi_n, \qquad (\operatorname{cd}\xi_n)'=-(1-m^2)\operatorname{sd}\xi_n\operatorname{nd}\xi_n, \qquad (\operatorname{nd}\xi_n)'=m^2\operatorname{cd}\xi_n\operatorname{sd}\xi_n.
\end{aligned}
$$

In addition we know that

$$
\operatorname{sn}(\xi_1+\xi_2)=\frac{\operatorname{sn}\xi_1\operatorname{cn}\xi_2\operatorname{dn}\xi_2+\operatorname{sn}\xi_2\operatorname{cn}\xi_1\operatorname{dn}\xi_1}{1-m^2\operatorname{sn}^2\xi_1\operatorname{sn}^2\xi_2}, \tag{5}
$$

$$
\operatorname{cn}(\xi_1+\xi_2)=\frac{\operatorname{cn}\xi_1\operatorname{cn}\xi_2-\operatorname{sn}\xi_1\operatorname{dn}\xi_1\operatorname{sn}\xi_2\operatorname{dn}\xi_2}{1-m^2\operatorname{sn}^2\xi_1\operatorname{sn}^2\xi_2}, \tag{6}
$$

$$
\operatorname{dn}(\xi_1+\xi_2)=\frac{\operatorname{dn}\xi_1\operatorname{dn}\xi_2-m^2\operatorname{sn}\xi_1\operatorname{cn}\xi_1\operatorname{sn}\xi_2\operatorname{cn}\xi_2}{1-m^2\operatorname{sn}^2\xi_1\operatorname{sn}^2\xi_2}. \tag{7}
$$

Thus, we further assume

$$
\begin{aligned}
\phi_{n+p_s}(\xi_n)&=a_0+a_1f_{n+p_s,i}+b_1g_{n+p_s,i}+c_1h_{n+p_s,i}+\sum_{j=2}^{l}f_{n+p_s,i}^{j-2}[a_jf_{n+p_s,i}^2\\
&\quad+b_jf_{n+p_s,i}g_{n+p_s,i}+c_jf_{n+p_s,i}h_{n+p_s,i}+d_jg_{n+p_s,i}h_{n+p_s,i}]\\
&\quad(i=1,2,3,4).
\end{aligned}
\tag{8}
$$

From (4) and the identities (5–7), one has

$$
\begin{aligned}
f_{n+p_s,1}(\xi_n) &= \frac{\operatorname{sn}\xi_n \operatorname{cn}\varphi_s \operatorname{dn}\varphi_s + \operatorname{sn}\varphi_s \operatorname{cn}\xi_n \operatorname{dn}\xi_n}{1 - m^2 \operatorname{sn}^2\xi_n \operatorname{sn}^2\varphi_s},\\
g_{n+p_s,1}(\xi_n) &= \frac{\operatorname{cn}\xi_n \operatorname{cn}\varphi_s - \operatorname{sn}\xi_n \operatorname{dn}\xi_n \operatorname{sn}\varphi_s \operatorname{dn}\varphi_s}{1 - m^2 \operatorname{sn}^2\xi_n \operatorname{sn}^2\varphi_s},\\
h_{n+p_s,1}(\xi_n) &= \frac{\operatorname{dn}\xi_n \operatorname{dn}\varphi_s - m^2 \operatorname{sn}\xi_n \operatorname{cn}\xi_n \operatorname{sn}\varphi_s \operatorname{cn}\varphi_s}{1 - m^2 \operatorname{sn}^2\xi_n \operatorname{sn}^2\varphi_s},\\
f_{n+p_s,2}(\xi_n) &= \frac{1 - m^2 \operatorname{sn}^2\xi_n \operatorname{sn}^2\varphi_s}{\operatorname{sn}\xi_n \operatorname{cn}\varphi_s \operatorname{dn}\varphi_s + \operatorname{sn}\varphi_s \operatorname{cn}\xi_n \operatorname{dn}\xi_n},\\
g_{n+p_s,2}(\xi_n) &= \frac{\operatorname{cn}\xi_n \operatorname{cn}\varphi_s - \operatorname{sn}\xi_n \operatorname{dn}\xi_n \operatorname{sn}\varphi_s \operatorname{dn}\varphi_s}{\operatorname{sn}\xi_n \operatorname{cn}\varphi_s \operatorname{dn}\varphi_s + \operatorname{sn}\varphi_s \operatorname{cn}\xi_n \operatorname{dn}\xi_n},\\
h_{n+p_s,2}(\xi_n) &= \frac{\operatorname{dn}\xi_n \operatorname{dn}\varphi_s - m^2 \operatorname{sn}\xi_n \operatorname{cn}\xi_n \operatorname{sn}\varphi_s \operatorname{cn}\varphi_s}{\operatorname{sn}\xi_n \operatorname{cn}\varphi_s \operatorname{dn}\varphi_s + \operatorname{sn}\varphi_s \operatorname{cn}\xi_n \operatorname{dn}\xi_n},\\
f_{n+p_s,3}(\xi_n) &= \frac{\operatorname{sn}\xi_n \operatorname{cn}\varphi_s \operatorname{dn}\varphi_s + \operatorname{sn}\varphi_s \operatorname{cn}\xi_n \operatorname{dn}\xi_n}{\operatorname{cn}\xi_n \operatorname{cn}\varphi_s - \operatorname{sn}\xi_n \operatorname{dn}\xi_n \operatorname{sn}\varphi_s \operatorname{dn}\varphi_s},\\
g_{n+p_s,3}(\xi_n) &= \frac{1 - m^2 \operatorname{sn}^2\xi_n \operatorname{sn}^2\varphi_s}{\operatorname{cn}\xi_n \operatorname{cn}\varphi_s - \operatorname{sn}\xi_n \operatorname{dn}\xi_n \operatorname{sn}\varphi_s \operatorname{dn}\varphi_s},\\
h_{n+p_s,3}(\xi_n) &= \frac{\operatorname{dn}\xi_n \operatorname{dn}\varphi_s - m^2 \operatorname{sn}\xi_n \operatorname{cn}\xi_n \operatorname{sn}\varphi_s \operatorname{cn}\varphi_s}{\operatorname{cn}\xi_n \operatorname{cn}\varphi_s - \operatorname{sn}\xi_n \operatorname{dn}\xi_n \operatorname{sn}\varphi_s \operatorname{dn}\varphi_s},\\
f_{n+p_s,4}(\xi_n) &= \frac{\operatorname{sn}\xi_n \operatorname{cn}\varphi_s \operatorname{dn}\varphi_s + \operatorname{sn}\varphi_s \operatorname{cn}\xi_n \operatorname{dn}\xi_n}{\operatorname{dn}\xi_n \operatorname{dn}\varphi_s - m^2 \operatorname{sn}\xi_n \operatorname{cn}\xi_n \operatorname{sn}\varphi_s \operatorname{cn}\varphi_s},\\
g_{n+p_s,4}(\xi_n) &= \frac{\operatorname{cn}\xi_n \operatorname{cn}\varphi_s - \operatorname{sn}\xi_n \operatorname{dn}\xi_n \operatorname{sn}\varphi_s \operatorname{dn}\varphi_s}{\operatorname{dn}\xi_n \operatorname{dn}\varphi_s - m^2 \operatorname{sn}\xi_n \operatorname{cn}\xi_n \operatorname{sn}\varphi_s \operatorname{cn}\varphi_s},\\
h_{n+p_s,4}(\xi_n) &= \frac{1 - m^2 \operatorname{sn}^2\xi_n \operatorname{sn}^2\varphi_s}{\operatorname{dn}\xi_n \operatorname{dn}\varphi_s - m^2 \operatorname{sn}\xi_n \operatorname{cn}\xi_n \operatorname{sn}\varphi_s \operatorname{cn}\varphi_s},
\end{aligned}
\tag{9}
$$

with φ_s satisfying

$$\varphi_s = p_{s1}d_1 + p_{s2}d_2 + \cdots + p_{sQ}d_Q. \tag{10}$$

Meanwhile, it is important to note that ϕ_{n+p_s} is a function of ξ_n and not ξ_{n+p_s}.

Step 3 Determine the degree of the polynomial solution (3) with the ansatz (8). We define the degree of u_{ξ_n} as $D[u_{\xi_n}] = l$, which gives rise to the degree of other expression as

$$D\left[\frac{d^k u}{d\xi_n^k}\right] = l + k, \qquad D\left[\left(\frac{d^k u}{d\xi_n^k}\right)^{\beta}\right] = \beta(l + k), \qquad D\left[u^{\alpha}\left(\frac{d^k u}{d\xi_n^k}\right)^{\beta}\right] = \alpha l + \beta(l + k).$$

Then l in (3) and (8) can be fixed by balancing the derivative term of the highest order with the highest nonlinear term in (2). Suppose we are interested in balancing terms with shift p_h, then the terms with shift order than p_h, say p_s, will not affect the balance since $f_{n+p_{s,i}}$ and $g_{n+p_{s,i}}$ can be interpreted as being of degree zero in $f_{n+p_{h,i}}$ and $g_{n+p_{h,i}}$.

Step 4 Substituting the ansatz (3) and (8) along with (4) and (9) into (2), then setting the coefficients of all powers like Jacobian elliptic functions to zero, we will get a series of algebraic equations, form which the constants a_0, a_j, b_j $(j = 1, 2, \dots, l)$ and c_j $(j = 1, 2, \dots, N)$ are explicitly determined.

Step 5 Substituting the obtained constants a_0, a_j, b_j $(j = 1, 2, \dots, l)$ and c_j $(j = 1, 2, \dots, N)$ back into the ansatz (3), we may get possible solutions of (1). Considering Jacobi elliptic functions may degenerate into hyperbolic functions and trigonometric functions when the modulus $\to 1$ and 0 respectively, therefore we also obtain many types of hyperbolic function solutions including soliton solutions and trigonometric function solutions of (1).

3 Exact Solutions for AL System

Now, applying the method developed above, we consider the Ablowitz–Ladik (AL) lattice system (also called the integrable discrete nonlinear Schrödinger equation)

$$i\frac{du_n}{dt} + \alpha(u_{n+1} + u_{n-1} - 2u_n) + \gamma|u_n|^2(u_{n+1} + u_{n-1}) = 0. \tag{11}$$

Equation (11) was proposed by using inverse scattering method [22]. The AL system has N-soliton solutions and a rich mathematical structure, and there are many works about it and some its modifications [17, 23–25].

For AL system (11), fist we make the traveling wave transformation

$$u_n = e^{i\theta_n}\phi_n(\xi_n), \qquad \theta_n = pn + qt + \delta, \qquad \xi_n = kn + ct + \zeta, \tag{12}$$

and

$$u_{n+1} = e^{i\theta_n}e^{ip}\phi_{n+1}(\xi_n), \qquad u_{n-1} = e^{i\theta_n}e^{-ip}\phi_{n-1}(\xi_n) \tag{13}$$

with $e^{\pm ip} = \cos(p) \pm i\sin(p)$, then (11) is reduced to

$$\begin{aligned} &-q\phi_n + \cos(p)(\alpha + \gamma\phi_n^2)(\phi_{n+1} + \phi_{n-1}) - 2\alpha\phi_n \\ &\quad + i[c\phi_n' + \sin(p)(\alpha + \gamma\phi_n^2)(\phi_{n+1} - \phi_{n-1})] = 0. \end{aligned} \tag{14}$$

Thus we have

$$-q\phi_n + \cos(p)(\alpha + \gamma\phi_n^2)(\phi_{n+1} + \phi_{n-1}) - 2\alpha\phi_n = 0, \tag{15}$$

$$c\phi_n' + \sin(p)(\alpha + \gamma\phi_n^2)(\phi_{n+1} - \phi_{n-1}) = 0. \tag{16}$$

We expand the solution of (15) and (16) in the forms of (3) and (8). Balancing the linear term of the highest order with the highest nonlinear term in (16), we determined $l = 1$, then the solution for AL system reads in the form

$$\phi_{n+p_s}(\xi_n) = a_0 + a_1 f_{n+p_s,i} + b_1 g_{n+p_s,i} + c_1 h_{n+p_s,i} \quad (p_s = -1, 0, 1;\ i = 1, 2, 3, 4). \tag{17}$$

3.1 $\operatorname{sn}\xi_n$, $\operatorname{cn}\xi_n$ and $\operatorname{dn}\xi_n$ Expansion

When $i = 1$ in ansatz (17), the solution of (15–16) is expanded by $\operatorname{sn}\xi_n$, $\operatorname{cn}\xi_n$ and $\operatorname{dn}\xi_n$. We have

$$\begin{aligned} \phi_n(\xi_n) &= a_0 + a_1 f_{n,1} + b_1 g_{n,1} + c_1 h_{n,1}, \\ \phi_{n+1}(\xi_n) &= a_0 + a_1 f_{n+1,1} + b_1 g_{n+1,1} + c_1 h_{n+1,1}, \\ \phi_{n-1}(\xi_n) &= a_0 + a_1 f_{n-1,1} + b_1 g_{n-1,1} + c_1 h_{n-1,1}, \end{aligned} \tag{18}$$

with

$$\begin{aligned}
&f_{n,1} = \operatorname{sn}\xi_n, \qquad g_{n,1} = \operatorname{cn}\xi_n, \qquad h_{n,1} = \operatorname{dn}\xi_n, \\
&f_{n+1,1} = \frac{\operatorname{sn}\xi_n \operatorname{cn}k \operatorname{dn}k + \operatorname{sn}k \operatorname{cn}\xi_n \operatorname{dn}\xi_n}{1 - m^2 \operatorname{sn}^2\xi_n \operatorname{sn}^2 k}, \\
&f_{n-1,1} = \frac{\operatorname{sn}\xi_n \operatorname{cn}k \operatorname{dn}k - \operatorname{sn}k \operatorname{cn}\xi_n \operatorname{dn}\xi_n}{1 - m^2 \operatorname{sn}^2\xi_n \operatorname{sn}^2 k}, \\
&g_{n+1,1} = \frac{\operatorname{cn}\xi_n \operatorname{cn}k - \operatorname{sn}\xi_n \operatorname{dn}\xi_n \operatorname{sn}k \operatorname{dn}k}{1 - m^2 \operatorname{sn}^2\xi_n \operatorname{sn}^2 k}, \\
&g_{n-1,1} = \frac{\operatorname{cn}\xi_n \operatorname{cn}k + \operatorname{sn}\xi_n \operatorname{dn}\xi_n \operatorname{sn}k \operatorname{dn}k}{1 - m^2 \operatorname{sn}^2\xi_n \operatorname{sn}^2 k}, \\
&h_{n+1,1} = \frac{\operatorname{dn}\xi_n \operatorname{dn}k - m^2 \operatorname{sn}\xi_n \operatorname{cn}\xi_n \operatorname{sn}k \operatorname{cn}k}{1 - m^2 \operatorname{sn}^2\xi_n \operatorname{sn}^2 k}, \\
&h_{n-1,1} = \frac{\operatorname{dn}\xi_n \operatorname{dn}k + m^2 \operatorname{sn}\xi_n \operatorname{cn}\xi_n \operatorname{sn}k \operatorname{cn}k}{1 - m^2 \operatorname{sn}^2\xi_n \operatorname{sn}^2 k}.
\end{aligned} \tag{19}$$

Substituting (18) with (19) into (15–16), clearing the denominator and setting the coefficients of all powers like $\operatorname{sn}^i\xi_n$ $(i = 0, 1, 2, 3, 4)$, $\operatorname{cn}\xi_n \operatorname{sn}^j\xi_n$ $(j = 0, 1, 2, 3)$, $\operatorname{dn}\xi_n \operatorname{sn}^i\xi_n$ $(i = 0, 1, 2, 3)$ and $\operatorname{cn}\xi_n \operatorname{dn}\xi_n \operatorname{sn}^j\xi_n$ $(j = 0, 1, 2)$ to zero, we have a system of algebraic equations respect to a_0, a_1, b_1, c_1, q and c. Solving the set of algebraic equations yields

$$\begin{aligned}
&a_0 = a_1 = c_1 = 0, \qquad b_1 = \pm m\sqrt{\frac{\alpha}{\gamma}}\operatorname{sd}(k), \\
&c = -2\alpha\sin(p)\operatorname{sd}(k), \qquad q = -2\alpha + 2\alpha\cos(p)\operatorname{cd}(k)\operatorname{nd}(k);
\end{aligned} \tag{20}$$

$$\begin{aligned}
&a_0 = b_1 = c_1 = 0, \qquad a_1 = \pm m\sqrt{-\frac{\alpha}{\gamma}}\operatorname{sn}(k), \\
&c = -2\alpha\sin(p)\operatorname{sn}(k), \qquad q = -2\alpha + 2\alpha\cos(p)\operatorname{cn}(k)\operatorname{dn}(k);
\end{aligned} \tag{21}$$

$$\begin{aligned}
&a_0 = a_1 = b_1 = 0, \qquad c_1 = \pm m\sqrt{\frac{\alpha}{\gamma}}\operatorname{sc}(k), \\
&c = -2\alpha\sin(p)\operatorname{sc}(k), \qquad q = -2\alpha + 2\alpha\cos(p)\operatorname{dc}(k)\operatorname{nc}(k),
\end{aligned} \tag{22}$$

where p and k are two arbitrary constants. Thus we obtain three doubly periodic solutions of AL system (11) as follows

$$\begin{aligned}
u_{1,n} = {} & \pm m\sqrt{\frac{\alpha}{\gamma}}\operatorname{sd}(k)\operatorname{cn}\{kn - 2\alpha\sin(p)\operatorname{sd}(k)t + \zeta\} \\
& \times \exp\{i[pn + (-2\alpha + 2\alpha\cos(p)\operatorname{cd}(k)\operatorname{nd}(k))t + \delta]\}, \quad \frac{\alpha}{\gamma} > 0,
\end{aligned} \tag{23}$$

$$\begin{aligned}
u_{2,n} = {} & \pm m\sqrt{-\frac{\alpha}{\gamma}}\operatorname{sn}(k)\operatorname{sn}\{kn - 2\alpha\sin(p)\operatorname{sn}(k)t + \zeta\} \\
& \times \exp\{i[pn + (-2\alpha + 2\alpha\cos(p)\operatorname{cn}(k)\operatorname{dn}(k))t + \delta]\}, \quad -\frac{\alpha}{\gamma} > 0,
\end{aligned} \tag{24}$$

$$u_{3,n} = \pm m\sqrt{\frac{\alpha}{\gamma}}\,\mathrm{sc}(k)\,\mathrm{dn}\{kn - 2\alpha\sin(p)\,\mathrm{sc}(k)t + \zeta\}$$
$$\times \exp\{i[pn + (-2\alpha + 2\alpha\cos(p)\,\mathrm{dc}(k)\,\mathrm{nc}(k))t + \delta]\}, \quad \frac{\alpha}{\gamma} > 0. \tag{25}$$

3.2 ns ξ_n, cs ξ_n and ds ξ_n Expansion

Similarly, when $i = 2$ in ansatz (17), the solution of (15–16) is expanded by ns ξ_n, cs ξ_n and ds ξ_n. Then we have

$$\begin{aligned} &a_0 = b_1 = c_1 = 0, \qquad a_1 = \pm\sqrt{-\frac{\alpha}{\gamma}}\,\mathrm{sn}(k), \\ &c = -2\alpha\sin(p)\,\mathrm{sn}(k), \qquad q = -2\alpha + 2\alpha\cos(p)\,\mathrm{cn}(k)\,\mathrm{dn}(k); \end{aligned} \tag{26}$$

$$\begin{aligned} &a_0 = a_1 = c_1 = 0, \qquad b_1 = \pm\sqrt{-\frac{\alpha}{\gamma}}\,\mathrm{sc}(k), \\ &c = -2\alpha\sin(p)\,\mathrm{sc}(k), \qquad q = -2\alpha + 2\alpha\cos(p)\,\mathrm{dc}(k)\,\mathrm{nc}(k); \end{aligned} \tag{27}$$

$$\begin{aligned} &a_0 = a_1 = b_1 = 0, \qquad c_1 = \pm\sqrt{-\frac{\alpha}{\gamma}}\,\mathrm{sd}(k), \\ &c = -2\alpha\sin(p)\,\mathrm{sd}(k), \qquad q = -2\alpha + 2\alpha\cos(p)\,\mathrm{cd}(k)\,\mathrm{nd}(k), \end{aligned} \tag{28}$$

where p and k are two arbitrary constants. The following doubly periodic wave solutions of AL system are derived correspondingly

$$u_{4,n} = \pm\sqrt{-\frac{\alpha}{\gamma}}\,\mathrm{sn}(k)\,\mathrm{ns}\{kn - 2\alpha\sin(p)\,\mathrm{sn}(k)t + \zeta\}$$
$$\times \exp\{i[pn + (-2\alpha + 2\alpha\cos(p)\,\mathrm{cn}(k)\,\mathrm{dn}(k))t + \delta]\}, \quad -\frac{\alpha}{\gamma} > 0, \tag{29}$$

$$u_{5,n} = \pm\sqrt{-\frac{\alpha}{\gamma}}\,\mathrm{sc}(k)\,\mathrm{cs}\{kn - 2\alpha\sin(p)\,\mathrm{sc}(k)t + \zeta\}$$
$$\times \exp\{i[pn + (-2\alpha + 2\alpha\cos(p)\,\mathrm{dc}(k)\,\mathrm{nc}(k))t + \delta]\}, \quad -\frac{\alpha}{\gamma} > 0, \tag{30}$$

$$u_{6,n} = \pm\sqrt{-\frac{\alpha}{\gamma}}\,\mathrm{sd}(k)\,\mathrm{ds}\{kn - 2\alpha\sin(p)\,\mathrm{sd}(k)t + \zeta\}$$
$$\times \exp\{i[pn + (-2\alpha + 2\alpha\cos(p)\,\mathrm{cd}(k)\,\mathrm{nd}(k))t + \delta]\}, \quad -\frac{\alpha}{\gamma} > 0. \tag{31}$$

3.3 sc ξ_n, nc ξ_n and dc ξ_n Expansion

When $i = 3$ in ansatz (17), the solution of (15–16) is expanded by sc ξ_n, nc ξ_n and dc ξ_n. Correspondingly, we have

$$\begin{aligned} &a_0 = b_1 = c_1 = 0, \qquad a_1 = \pm\sqrt{-\frac{\alpha(1-m^2)}{\gamma}}\,\mathrm{sc}(k), \\ &c = -2\alpha\sin(p)\,\mathrm{sc}(k), \qquad q = -2\alpha + 2\alpha\cos(p)\,\mathrm{dc}(k)\,\mathrm{nc}(k); \end{aligned} \tag{32}$$

$$a_0 = a_1 = c_1 = 0, \qquad b_1 = \pm\sqrt{-\frac{\alpha(1-m^2)}{\gamma}}\,\mathrm{sd}(k),$$
$$c = -2\alpha\sin(p)\frac{\mathrm{sn}(k)}{\mathrm{dn}(k)}, \qquad q = -2\alpha + 2\alpha\cos(p)\,\mathrm{cd}(k)\,\mathrm{nd}(k); \tag{33}$$

$$a_0 = a_1 = b_1 = 0, \qquad c_1 = \pm\sqrt{-\frac{\alpha}{\gamma}}\,\mathrm{sn}(k),$$
$$c = -2\alpha\sin(p)\,\mathrm{sn}(k), \qquad q = -2\alpha + 2\alpha\cos(p)\,\mathrm{cn}(k)\,\mathrm{dn}(k), \tag{34}$$

where p and k are two arbitrary constants, then we obtain three doubly periodic solutions of AL system as follows

$$u_{7,n} = \pm\sqrt{-\frac{\alpha(1-m^2)}{\gamma}}\,\mathrm{sc}(k)\,\mathrm{sc}\{kn + -2\alpha\sin(p)\,\mathrm{sc}(k)t + \zeta\}$$
$$\times \exp\{i[pn + (-2\alpha + 2\alpha\cos(p)\,\mathrm{dc}(k)\,\mathrm{nc}(k))t + \delta]\}, \tag{35}$$

$$u_{8,n} = \pm\sqrt{-\frac{\alpha(1-m^2)}{\gamma}}\,\mathrm{sd}(k)\,\mathrm{nc}\{kn - 2\alpha\sin(p)\,\mathrm{sd}(k)t + \zeta\}$$
$$\times \exp\{i[pn + (-2\alpha + 2\alpha\cos(p)\,\mathrm{cd}(k)\,\mathrm{nd}(k))t + \delta]\}, \tag{36}$$

$$u_{9,n} = \pm\sqrt{-\frac{\alpha}{\gamma}}\,\mathrm{sn}(k)\,\mathrm{dc}\{kn - 2\alpha\sin(p)\,\mathrm{sn}(k)t + \zeta\}$$
$$\times \exp\{i[pn + (-2\alpha + 2\alpha\cos(p)\,\mathrm{cn}(k)\,\mathrm{dn}(k))t + \delta]\}, \tag{37}$$

where $-\alpha/\gamma > 0$.

3.4 $\mathrm{sd}\,\xi_n$, $\mathrm{cd}\,\xi_n$ and $\mathrm{nd}\,\xi_n$ Expansion

When $i = 4$ in ansatz (17), the solution of (15–16) is expanded by $\mathrm{sd}\,\xi_n$, $\mathrm{cd}\,\xi_n$ and $\mathrm{nd}\,\xi_n$. Correspondingly, we have

$$a_0 = b_1 = c_1 = 0, \qquad a_1 = \pm m\sqrt{-\frac{\alpha(1-m^2)}{\gamma}}\,\mathrm{sd}(k),$$
$$c = -2\alpha\sin(p)\,\mathrm{sd}(k), \qquad q = -2\alpha + 2\alpha\cos(p)\,\mathrm{cd}(k)\,\mathrm{nd}(k); \tag{38}$$

$$a_0 = a_1 = c_1 = 0, \qquad b_1 = \pm m\sqrt{-\frac{\alpha}{\gamma}}\,\mathrm{sn}(k),$$
$$c = -2\alpha\sin(p)\,\mathrm{sn}(k), \qquad q = -2\alpha + 2\alpha\cos(p)\,\mathrm{cn}(k)\,\mathrm{dn}(k); \tag{39}$$

$$a_0 = a_1 = b_1 = 0, \qquad c_1 = \pm\sqrt{-\frac{\alpha(1-m^2)}{\gamma}}\,\mathrm{sc}(k),$$
$$c = -2\alpha\sin(p)\,\mathrm{sc}(k), \qquad q = -2\alpha + 2\alpha\cos(p)\,\mathrm{dc}(k)\,\mathrm{nc}(k), \tag{40}$$

Table 1

m	$\mathrm{sn}\,\xi_n$	$\mathrm{cn}\,\xi_n$	$\mathrm{dn}\,\xi_n$	$\mathrm{sc}\,\xi_n$	$\mathrm{sd}\,\xi_n$	$\mathrm{cd}\,\xi_n$	$\mathrm{ns}\,\xi_n$	$\mathrm{nc}\,\xi_n$	$\mathrm{nd}\,\xi_n$	$\mathrm{cs}\,\xi_n$	$\mathrm{ds}\,\xi_n$	$\mathrm{dc}\,\xi_n$
$m \to 1$	$\tanh\xi_n$	$\mathrm{sech}\,\xi_n$	$\mathrm{sech}\,\xi_n$	$\sinh\xi_n$	$\sinh\xi_n$	1	$\coth\xi_n$	$\cosh\xi_n$	$\cosh\xi_n$	$\mathrm{csch}\,\xi_n$	$\mathrm{csch}\,\xi_n$	1
$m \to 0$	$\sin\xi_n$	$\cos\xi_n$	1	$\tan\xi_n$	$\sin\xi_n$	$\cos\xi_n$	$\csc\xi_n$	$\sec\xi_n$	1	$\cot\xi_n$	$\csc\xi_n$	$\sec\xi_n$

where p and k are two arbitrary constants, then we obtain the following solutions for AL system

$$u_{10,n} = \pm m\sqrt{-\frac{\alpha(1-m^2)}{\gamma}}\,\mathrm{sd}(k)\,\mathrm{sd}\{kn - 2\alpha\sin(p)\,\mathrm{sd}(k)t + \zeta\} + \exp\{i[pn + (-2\alpha + 2\alpha\cos(p)\,\mathrm{cd}(k)\,\mathrm{nd}(k))t + \delta]\}, \tag{41}$$

$$u_{11,n} = \pm m\sqrt{-\frac{\alpha}{\gamma}}\,\mathrm{sn}(k)\,\mathrm{cd}\{kn - 2\alpha\sin(p)\,\mathrm{sn}(k)t + \zeta\} \times \exp\{i[pn + (-2\alpha + 2\alpha\cos(p)\,\mathrm{cn}(k)\,\mathrm{dn}(k))t + \delta]\}, \tag{42}$$

$$u_{12,n} = \pm\sqrt{-\frac{\alpha(1-m^2)}{\gamma}}\,\mathrm{sc}(k)\,\mathrm{nd}\{kn - 2\alpha\sin(p)\,\mathrm{sc}(k)t + \zeta\} \times \exp\{i[pn + (-2\alpha + 2\alpha\cos(p)\,\mathrm{dc}(k)\,\mathrm{nc}(k))t + \delta]\}, \tag{43}$$

where $-\alpha/\gamma > 0$.

Above, we obtain twelve exact and explicit Jacobian elliptic function doubly periodic traveling wave solutions for AL system (11). When $\alpha = 1$ and $\gamma = \epsilon = \pm 1$, all the solutions reported in literature [17] are derived here with less algebraic expansion computations. And solutions $u_{9,n}$ (37) and $u_{11,n}$ (42) are new. As we know, Jacobian elliptic functions can be degenerated as hyperbolic functions when the modulus $m \to 1$ and trigonometric functions when $m \to 0$, which is shown in detail in Table 1.

So the doubly periodic wave solutions $u_{i,n}$ $(i = 1, 2, \ldots, 12)$ can be easily generated into hyperbolic function solutions, trigonometric function solutions and even constant amplitude envelope wave solutions respectively. Especially, when the modulus $m \to 1$, some Jacobian elliptic function solutions can be degenerate into solitonic solutions. Here, we omit listing all the degenerated solutions for simplification, and concern the soliton solutions only.

For instance, as for the doubly periodic wave solution $u_{1,n}$ expressed by (23), in the long wave limit $m \to 1$, we may obtain the bright soliton solution

$$u'_{1,n} = \sqrt{\frac{\alpha}{\gamma}}\sinh(k)\mathrm{sech}\,\{kn - 2\alpha\sin(p)\sinh(k)t + \zeta\} \times \exp\{i[pn + (-2\alpha + 2\alpha\cos(p)\cosh(k))t + \delta]\}, \quad \frac{\alpha}{\gamma} > 0. \tag{44}$$

The amplitude and the width of the solitons are fixed and defined by the parameters of the ansatz. Figure 1 gives the motion of discrete bright soliton for $|u'_{1,n}|$ with fixed n. The constant $\xi(\delta)$ in (44) is an arbitrary real constant, indicating translational invariance along the lattice. Although is seems simple, the translational invariance is not as trivial as in the case of the continuous equation. When $\xi(\delta)$ is zero, the soliton center is located between the lattice sites. Then the soliton shape is asymmetric. In this sense, the parameter $\xi(\delta)$ produces a continuous family of solitons with variable shape, see Fig. 2.

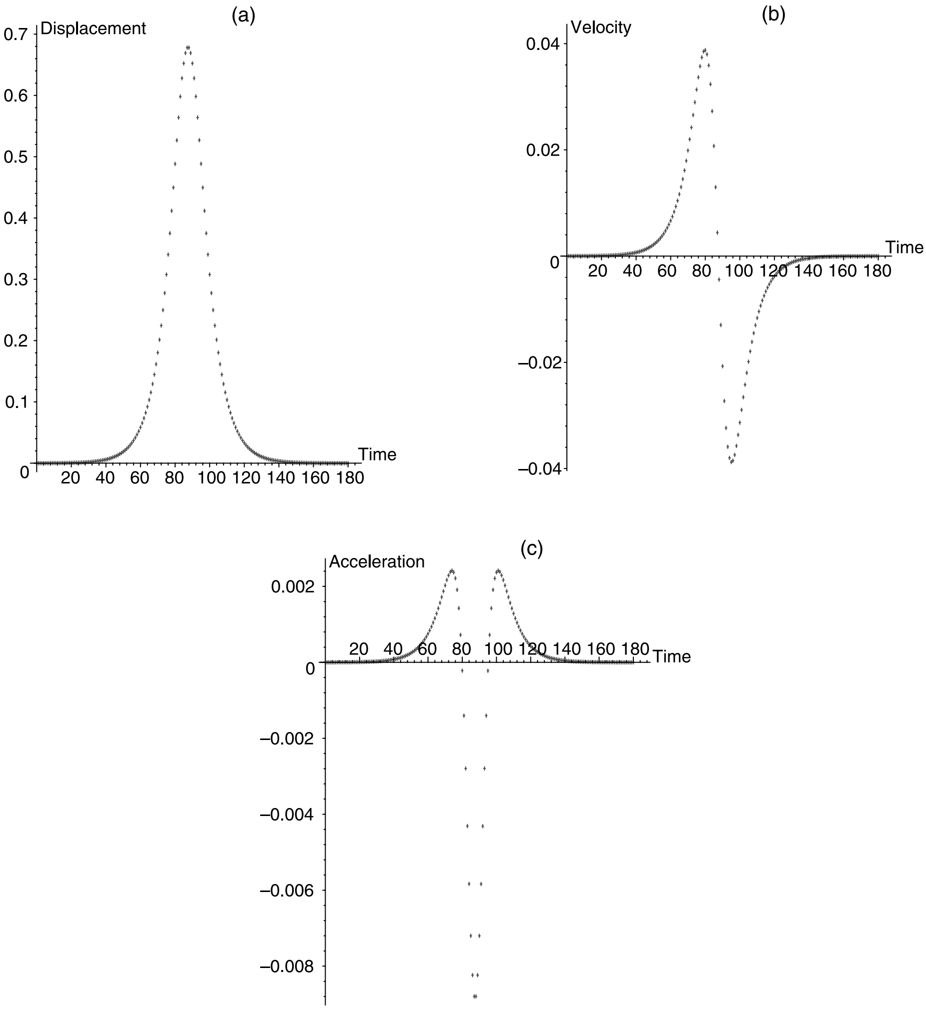

Fig. 1 Motion of discrete bright soliton for $|u'_{1,n}|$ with fixed n and parameters $\alpha = 1/5$, $\gamma = 1/20$, $k = 1/3$, $p = 1$, $\xi = 10 - n/3$, where (**a**) displacement, (**b**) velocity, (**c**) acceleration, respectively

For the doubly periodic wave solution $u_{2,n}$ expressed by (24), in the long wave limit $m \to 1$, the black soliton solution can be derived in the following

$$u'_{2,n} = \sqrt{-\frac{\alpha}{\gamma}} \tanh(k) \tanh(kn - 2\alpha \sin(p) \tanh(k)t + \zeta)$$
$$\times \exp\{i[pn + (-2\alpha + 2\alpha \cos(p)\mathrm{sech}^2(k))t + \delta]\}, \quad -\frac{\alpha}{\gamma} > 0. \tag{45}$$

Similar to the discrete bright soliton, the parameter $\xi(\delta)$ produces a continuous family of dark solitons with variable shape, see Fig. 3. In the limit of large n, the solutions (45) reduce to constants. This can be considered as another independent ('plane wave') solution of the AL system.

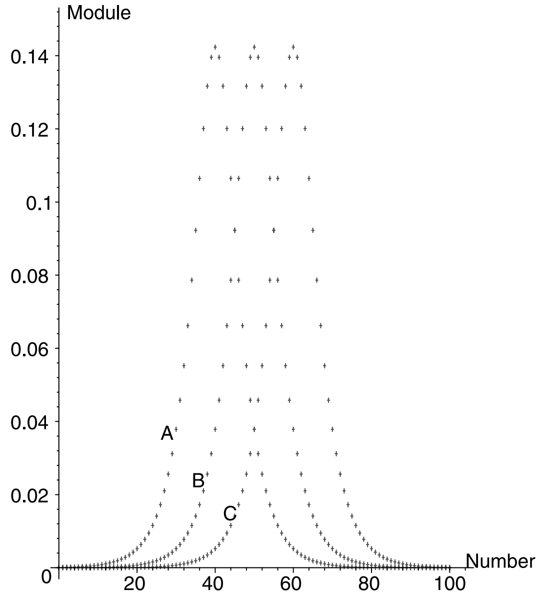

Fig. 2 Shape of discrete bright soliton for $|u'_{1,n}|$ with different phase ζ and parameters $\alpha = 1/10$, $\gamma = -1/5$, $k = 1/5$, $p = 1$ at $t = 0$, where $\zeta = -8$ for A, $\zeta = -10$ for B, $\zeta = -12$ for C

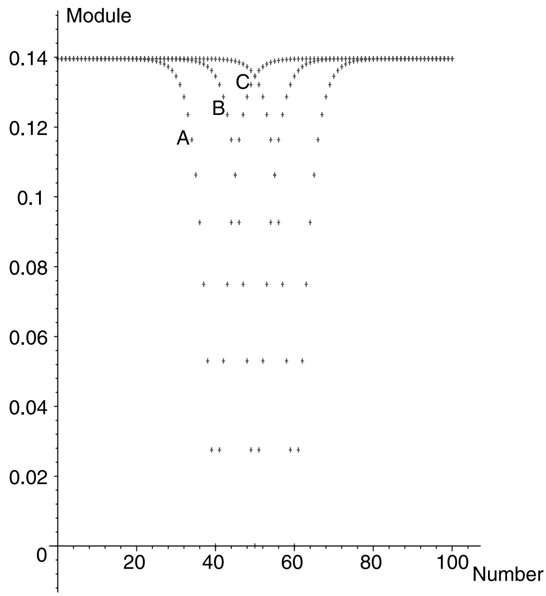

Fig. 3 Shape of discrete dark soliton for $|u'_{2,n}|$ with different phase ζ and parameters $\alpha = 1/10$, $\gamma = -1/5$, $k = 1/5$, $p = 1$ at $t = 0$, where $\zeta = -8$ for A, $\zeta = -10$ for B, $\zeta = -12$ for C

4 Summary

In summary, the general Jacobi elliptic function expansion algorithm for continuous NPDEs is developed and extended to construct abundant doubly periodic wave solutions for discrete nonlinear equations. Though the modification on the elliptic function expansion algorithm is slight, it is important. Applying this method, twelve doubly periodic traveling wave solutions for the AL system are obtained. In limit conditions, when the modulus $m \to 1$ or $m \to 0$, these solutions degenerate into hyperbolic function solutions and trigonometric function solutions respectively. Some solution are new to our knowledge. Especially, in the long wave limit $m \to 1$, some of Jacobi elliptic function solutions may degenerate as solitary wave solutions including bright soliton and dark soliton as well. And for the soliton solutions, interesting and important properties are revealed. Similar to the other integrable systems, it is likely that the exact solutions of the AL system in this paper are stable. The more to apply this method to other nonlinear discrete models and explore the solutions for the AL system is need to further study.

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